

The Universal Pythagorean Angle Lattice: Converging to π , e , γ , $\zeta(3)$, Catalan's Constant, and the Golden Ratio φ

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Abstract

We introduce the *Pythagorean angle lattice*, a countable set of real numbers obtained by applying the cotangent function to rational divisions of finite sums of acute angles of primitive Pythagorean triples. We prove that a distinguished single-partner-angle sublattice of this structure is *constructively universal*: for every positive real constant $T > 0$, and for any starting primitive Pythagorean triple with larger acute angle α_0 , there exists an explicit sequence of primitive Pythagorean triples such that

$$\cot\left(\frac{\alpha_0 + \phi(u_k, v_k)}{D}\right) \rightarrow T,$$

where D is an integer denominator determined by T and α_0 , and $\phi(u_k, v_k)$ is the Euclid angle of the partner triple. The sequence is generated by the continued fraction convergents (with a parity correction using semiconvergents) of an explicit target ratio $t = \tan((\text{Darccot}(T) - \alpha_0)/2)$. The convergence rate is $O(1/u_k^2)$ for subsequences of true convergents, and $O(1/u_k)$ for semiconvergents.

We demonstrate the universality of the construction with five pillar constants: π , Euler's number e , the Euler–Mascheroni constant γ , Apéry's constant $\zeta(3)$, and Catalan's constant G , and we establish the golden ratio φ as an exact special case. The framework unifies algebraic constants (metallic means and the golden ratio φ), transcendental constants (π, e), and constants whose irrationality remains open ($\gamma, \zeta(3), G$), within a single geometric construction arising from integer right triangles. The general multi-angle lattice is left for future investigation.

Keywords: Pythagorean triples, continued fractions, metallic means, golden ratio, universal approximation, π , e , Euler–Mascheroni constant, Apéry's constant, Catalan's constant, cotangent, primitive triples, Diophantine approximation.

1. Introduction

The metallic means, introduced by de Spinadel [1], are the positive roots of $x^2 - mx - 1 = 0$:

$$M_m = \frac{m + \sqrt{m^2 + 4}}{2}.$$

In recent work [2], a connection was established between these metallic means and primitive Pythagorean triples. For a primitive Pythagorean triple (a, b, c) with $c - b \in \{1, 2, 8\}$,

$$\cot\left(\frac{\theta}{4}\right) = M_n,$$

where $n = 2a/(c - b)$, θ is the smaller acute angle, and M_n is the n -th metallic mean. This identity is universally applicable for metallic means of order $n \geq 5$. The cases $n = 1, 2, 3, 4$ require separate treatment [2]. This connected the metallic means – an

algebraic family of constants – to the geometry of right triangles. A closely related cotangent normalization yields the Golden ratio

$$\varphi = \frac{1 + \sqrt{5}}{2}$$

exactly, for *every* primitive Pythagorean triple (a, b, c) with $a < b < c$, with no restriction whatsoever, when composed with the fundamental $(3, 4, 5)$ triple via the Brahmagupta–Fibonacci identity: writing $\alpha_0 = \arctan(b/a)$ for the larger acute angle of (a, b, c) , (a', b', c') for the composed triple with $a' = |3a - 4b|$, $b' = 4a + 3b$, and

$$\beta(a', b') := \arctan\left(\frac{b'}{a'}\right)$$

for the corresponding acute angle of the composed triangle (the larger one if $b' > a'$, the smaller one if $a' > b'$),

$$\cot\left(\frac{\alpha_0 + \beta(a', b')}{4}\right) = \varphi.$$

We prove this identity in full generality, unconditionally, as Proposition 3.5 in Section 3. (The condition $b/a < 7$ classifies whether $\beta(a', b')$ is the larger or smaller acute angle of the composed triple; the identity itself requires no such restriction.) We emphasize that $\beta(a', b')$ is the *ordinary* acute angle of triangle (a', b', c') and, despite its superficial resemblance, is *not* generally the same as the Euclid angle $\phi(u', v')$ of Definition 2.3 applied to the Euclid parameters of (a', b', c') ; see Remark 3.6 for the precise relationship.

This observation naturally raises the question whether the same construction can be extended to a transcendental target. The present author answered this question in the affirmative in an earlier preprint [3]. For every primitive Pythagorean triple, there exists an explicit, provably convergent sequence of partner triples such that

$$\cot\left(\frac{\alpha_0 + \phi(u_k, v_k)}{4}\right) \rightarrow \frac{\pi}{2}.$$

In the present paper, we go further. We show that the construction is *universal*: for any positive real constant $T > 0$, there exists an integer denominator D such that an infinite sequence of primitive Pythagorean triples satisfies

$$\cot\left(\frac{\alpha_0 + \phi(u_k, v_k)}{D}\right) \rightarrow T.$$

The sequence is generated explicitly by the continued fraction convergents (with a parity correction using semiconvergents) of the target ratio

$$t = \tan\left(\frac{D \arccot(T) - \alpha_0}{2}\right).$$

We prove that the convergence rate is $O(1/u_k^2)$ for subsequences of true convergents, and $O(1/u_k)$ when semiconvergents are used.

We demonstrate this universality with five pillar constants: π , Euler's number e , the Euler–Mascheroni constant γ , Apéry's constant $\zeta(3)$, and Catalan's constant G . For each, we provide explicit convergence tables.

The paper is organized as follows. Section 2 collects the necessary preliminaries on primitive Pythagorean triples, the Euclid parametrization, the formal definition of the Pythagorean angle lattice, and continued fractions [5]. Section 3 presents the generalised construction, proves the main theorem, and establishes the golden ratio and

metallic means as two exact special cases of the general framework. Section 4 provides numerical verification for the five pillar constants. Section 5 concludes with a summary and open problems.

2. Preliminaries

2.1 Primitive Pythagorean Triples and Euclid Parametrization

A primitive Pythagorean triple is an ordered triple (a, b, c) of positive integers satisfying

$$a^2 + b^2 = c^2, \quad \gcd(a, b, c) = 1, \quad a < b < c.$$

By the classical Euclid parametrization [4, 8], there exist coprime integers $u > v > 0$ of opposite parity such that

$$a = u^2 - v^2, \quad b = 2uv, \quad c = u^2 + v^2.$$

We call (u, v) the Euclid parameters of the triple.

Remark 2.1. In the numerical tables of Section 4, triples are listed in the order produced by the Euclid map $(u^2 - v^2, 2uv, u^2 + v^2)$. This may not satisfy the ordering $a < b < c$; the mathematical content is unaffected.

Definition 2.2 (Larger acute angle). For a primitive Pythagorean triple (a, b, c) , define

$$\alpha(a, b) = \arctan\left(\frac{\max(a, b)}{\min(a, b)}\right) \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right).$$

This is always the larger of the two acute angles of the triangle, since $a \neq b$ for a primitive triple.

Definition 2.3 (Euclid angle map). For coprime $u > v > 0$ of opposite parity, define

$$\phi(u, v) = 2\arctan\left(\frac{v}{u}\right) \in \left(0, \frac{\pi}{2}\right).$$

Remark 2.4. As v/u ranges over $(0, 1)$, $\phi(u, v)$ ranges continuously and monotonically over $(0, \pi/2)$. The map $v/u \mapsto \phi(u, v)$ is a bijection from $(0, 1)$ onto $(0, \pi/2)$.

Important note: $\phi(u, v)$ equals the *larger* acute angle of the corresponding triple only when $v/u > \sqrt{2} - 1 \approx 0.4142$. When $v/u < \sqrt{2} - 1$, $\phi(u, v)$ is the *smaller* acute angle. Throughout this paper, we refer to $\phi(u, v)$ as the *Euclid angle* of the triple to avoid this ambiguity.

2.2 The Pythagorean Angle Lattice

Definition 2.5 (Pythagorean angle lattice). Let $\mathcal{T}$ be the set of all primitive Pythagorean triples with Euclid parameters (u, v) satisfying $\gcd(u, v) = 1$ and $u \not\equiv v \pmod{2}$. Define

$$\mathcal{L} = \left\{ \cot\left(\frac{\phi(u_0, v_0) + \phi(u_1, v_1) + \cdots + \phi(u_n, v_n)}{D}\right) : (u_i, v_i) \in \mathcal{T}, D \in \mathbb{N}, n \in \mathbb{N} \right\}.$$

This is the *Pythagorean angle lattice*.

In this paper, we fix one starting triple (a_0, b_0, c_0) with Euclid parameters (u_0, v_0) and consider the subset

$$\mathcal{L}_{\alpha_0} = \left\{ \cot\left(\frac{\alpha_0 + \phi(u, v)}{D}\right) : \gcd(u, v) = 1, u \not\equiv v \pmod{2}, D \in \mathbb{N} \right\}.$$

The constructive universality proven in this paper applies to $\mathcal{L}_{\alpha_0}$. The general multi-angle lattice $\mathcal{L}$ is left for future investigation.

2.3 Continued Fractions

Let $t \in (0, 1)$ be irrational with continued fraction expansion

$$t = [0; a_1, a_2, a_3, \dots].$$

Define $p_{-1} = 1, p_0 = 0, q_{-1} = 0, q_0 = 1$, and recursively

$$p_k = a_k p_{k-1} + p_{k-2}, \quad q_k = a_k q_{k-1} + q_{k-2}.$$

The convergents p_k/q_k satisfy $\gcd(p_k, q_k) = 1$, $p_k/q_k \rightarrow t$, and

$$\left| t - \frac{p_k}{q_k} \right| < \frac{1}{q_k q_{k+1}} = O\left(\frac{1}{q_k^2}\right).$$

For the theory of continued fractions, see [5].

2.4 The Parity Condition and Semiconvergents

For a pair (p, q) to serve as Euclid parameters (u, v) of a primitive Pythagorean triple, it must satisfy:

1. $\gcd(p, q) = 1$;
2. p and q have opposite parity (i.e., $p + q$ is odd).

Classical continued fractions guarantee coprimality but not opposite parity. The following lemma repairs this obstruction.

Lemma 2.6 (Admissible approximants exist infinitely often). There are infinitely many true convergents p_k/q_k with $p_k + q_k$ odd. Consequently, there are infinitely many coprime, opposite-parity pairs converging to t . Moreover, semiconvergents provide a practical method for finding admissible pairs with smaller denominators when a true convergent fails the parity test.

Proof. Let $S_n = p_n + q_n$. From the determinant identity $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$, reducing modulo 2 gives

$$p_n q_{n-1} + p_{n-1} q_n \equiv 1 \pmod{2}.$$

If both S_n and S_{n-1} were even, then all four of $p_n, q_n, p_{n-1}, q_{n-1}$ would be odd (coprimality excludes them being all even), making the left-hand side $1 \cdot 1 + 1 \cdot 1 \equiv 0 \pmod{2}$, a contradiction. Hence S_n and S_{n-1} cannot both be even. Since a sequence of integers with no two consecutive evens must contain infinitely many odds, infinitely many S_n are odd. Therefore, infinitely many true convergents are admissible.

When a true convergent fails the parity test, semiconvergents

$$(p_{k-2} + mp_{k-1}, q_{k-2} + mq_{k-1}), \quad m = 1, \dots, a_k,$$

can be used to find an admissible pair with smaller denominator. This is the practical role of semiconvergents in the numerical examples of Section 4. For the error bound $O(1/u_k)$ for semiconvergents, see [5].

3. The Generalised Construction

3.1 The Main Theorem

Theorem 3.1 (Generalised π -thagoras). Let $T > 0$ be any positive real number. Let (a_0, b_0, c_0) be any primitive Pythagorean triple with larger acute angle α_0 . Then there exists an integer $D \in \mathbb{N}$ such that:

$$0 < \alpha^* = \text{Darccot}(T) - \alpha_0 < \frac{\pi}{2}.$$

Define

$$t = \tan\left(\frac{\alpha^*}{2}\right) \in (0, 1).$$

Let (u_k, v_k) be the sequence of admissible (coprime, opposite-parity) continued fraction convergents (or semiconvergents) of t guaranteed by Lemma 2.6, and let (a_k, b_k, c_k) be the corresponding primitive Pythagorean triples with Euclid angles $\phi(u_k, v_k)$. Then

$$\cot\left(\frac{\alpha_0 + \phi(u_k, v_k)}{D}\right) \rightarrow T \quad \text{as } k \rightarrow \infty.$$

Moreover, the error satisfies:

$$\left| \cot\left(\frac{\alpha_0 + \phi(u_k, v_k)}{D}\right) - T \right| = \begin{cases} O(1/u_k^2), & \text{if } (u_k, v_k) \text{ is a true convergent,} \\ O(1/u_k), & \text{if } (u_k, v_k) \text{ is a semiconvergent.} \end{cases}$$

Proof. Let $\theta = \operatorname{arccot}(T) = \arctan(1/T) \in (0, \pi/2)$. Since $\theta \in (0, \pi/2)$, the interval $(\alpha_0, \alpha_0 + \pi/2)$ has length $\pi/2 > \theta$. Therefore, the sequence $D\theta$ for $D = 1, 2, 3, \dots$ must land in this interval for some D . The smallest such D is given by $D = \lfloor \alpha_0/\theta \rfloor + 1$. Hence $\alpha^* = D\theta - \alpha_0 \in (0, \pi/2)$.

Since $t = \tan(\alpha^*/2) \in (0, 1)$, by Lemma 2.6 there exists a sequence of admissible Euclid pairs (u_k, v_k) with $v_k/u_k \rightarrow t$. The continuity of $\phi(u, v) = 2\arctan(v/u)$ gives

$$\phi(u_k, v_k) \rightarrow 2\arctan(t) = \alpha^*.$$

Therefore,

$$\frac{\alpha_0 + \phi(u_k, v_k)}{D} \rightarrow \frac{\alpha_0 + \alpha^*}{D} = \theta.$$

Since $\cot$ is continuous at θ and $\cot\theta = T$, we get the desired limit.

For the error bound, from continued fraction theory, for true convergents:

$$\left| \frac{v_k}{u_k} - t \right| = O\left(\frac{1}{u_k^2}\right).$$

For semiconvergents selected by Lemma 2.6, the bound is generally

$$\left| \frac{v_k}{u_k} - t \right| = O\left(\frac{1}{u_k}\right),$$

as the quadratic bound is not guaranteed for semiconvergents other than the terminal one in each block (see [5]).

Since $\arctan$ has bounded derivative on $[0, 1]$,

$$|\phi(u_k, v_k) - \alpha^*| = 2|\arctan(v_k/u_k) - \arctan(t)| = \begin{cases} O(1/u_k^2), & \text{for true convergents,} \\ O(1/u_k), & \text{for semiconvergents.} \end{cases}$$

The function $x \mapsto \cot((\alpha_0 + x)/D)$ is differentiable with bounded derivative in a neighbourhood of α^* (since $\theta \in (0, \pi/2)$ is bounded away from the singularities of $\cot$). Hence the mean value theorem gives the claimed bound. The implied constants depend on T and degrade as $T \rightarrow 0^+$ or $T \rightarrow \infty$.

Corollary 3.2. For every positive real constant T , and for every starting primitive Pythagorean triple, the construction above produces an explicit sequence of partner triples achieving arbitrarily many correct decimal digits of T . The number of correct digits grows linearly in k for subsequences of true convergents.

3.2 The Algorithmic Construction

The proof of Theorem 3.1 is constructive. The following procedure summarises the algorithm:

1. Input $T > 0$ and starting triple (a_0, b_0, c_0) .
2. Compute $\alpha_0 = \arctan(\max(a_0, b_0)/\min(a_0, b_0))$.
3. Set $\theta = \operatorname{arccot}(T) = \arctan(1/T)$.
4. Compute $D = \lfloor \alpha_0/\theta \rfloor + 1$.
5. Compute $\alpha^* = D\theta - \alpha_0$ and $t = \tan(\alpha^*/2)$.
6. Generate the continued fraction convergents p_k/q_k of t .
7. For each convergent p_k/q_k :
 - If $p_k + q_k$ is odd, set $(u_k, v_k) = (p_k, q_k)$.
 - Otherwise, search the semiconvergents $p_{k-2} + mp_{k-1}$, $q_{k-2} + mq_{k-1}$ for the first $m \in \{1, \dots, a_k\}$ with odd sum.
8. Compute $(a_k, b_k, c_k) = (u_k^2 - v_k^2, 2u_kv_k, u_k^2 + v_k^2)$.

9. Compute $\phi(u_k, v_k) = 2\arctan(v_k/u_k)$.
10. Output $\cot((\alpha_0 + \phi(u_k, v_k))/D)$.

The termination of each step is guaranteed by Lemma 2.6. The k -th admissible pair is found in $O(k)$ continued-fraction steps: since Lemma 2.6's argument shows no two consecutive convergents can both fail the parity test, at most half of the first $2k$ convergents can be inadmissible, so the k -th admissible pair occurs among the first $2k$ convergents, giving the $O(k)$ bound directly.

3.3 The Metallic Means as a Special Case

The metallic-mean identity from [2] is recovered as a special exact (non-limiting) case of the framework. When T is such that $t = \tan(\alpha^*/2)$ is a rational number exactly realizable by a single primitive triple, and $D = 4$, the construction terminates in one step.

Specifically, for a primitive triple (a, b, c) with $c - b \in \{1, 2, 8\}$, the governing parameter $n = 2a/(c - b)$ gives:

$$\cot\left(\frac{\theta}{4}\right) = \frac{n + \sqrt{n^2 + 4}}{2} = M_n.$$

This identity holds for all $n \geq 5$. The cases $n = 1, 2, 3, 4$ are exceptional and are handled separately in [2].

Thus, the framework contains the metallic means as exact algebraic points embedded within a continuum of transcendental limits.

3.4 The Golden Ratio as a Second Special Case

The introductory example involving the golden ratio φ (obtained by scaling a primitive triple by the fundamental $(3, 4, 5)$ triple via the Brahmagupta–Fibonacci identity) is not an isolated numerical curiosity: it is a genuine closed-form identity that holds for *every* primitive Pythagorean triple with $a < b < c$, with no restriction whatsoever. We state and prove this precisely here.

Definition 3.3 (Ratio angle). For positive reals x, y , define $\beta(x, y) = \arctan(y/x) \in (0, \pi/2)$. Note that for the starting triple, $\alpha_0 = \beta(a, b)$, since $b > a$.

Lemma 3.4 (A golden-ratio angle identity). Let $\varphi = (1 + \sqrt{5})/2$ and $\theta = \operatorname{arccot}\varphi$. Then

$$\tan(2\theta) = 2, \quad \tan(4\theta) = -\frac{4}{3}, \quad 4\operatorname{arccot}\varphi = \pi - \arctan\left(\frac{4}{3}\right).$$

Proof. Let $t = \tan\theta = 1/\varphi$. Since $\varphi^2 = \varphi + 1$, dividing by φ^2 gives $1 = 1/\varphi + 1/\varphi^2$, i.e. $t^2 + t - 1 = 0$, so $t^2 = 1 - t$. By the double-angle formula, $\tan(2\theta) = 2t/(1 - t^2) = 2t/t = 2$. Applying the double-angle formula again gives $\tan(4\theta) = 4/(-3) = -4/3$. Since $\theta \in (0, \pi/4)$, we have $2\theta \in (0, \pi/2)$ and $\tan(2\theta) = 2 > 0$, so $2\theta = \arctan 2$. Then $4\theta \in (0, \pi)$, and since $\tan(4\theta) < 0$, necessarily $4\theta \in (\pi/2, \pi)$, giving $4\theta = \pi - \arctan(4/3)$.

Proposition 3.5 (Golden ratio from Pythagorean angle composition). Let (a, b, c) be *any* primitive Pythagorean triple with $a < b < c$ and $\alpha_0 = \beta(a, b)$. Let

$$a' = |3a - 4b|, \quad b' = 4a + 3b,$$

so that (a', b', c') with $c' = \sqrt{a'^2 + b'^2}$ is the triple obtained by the Gaussian-integer product $(a + bi)(3 + 4i) = -a' + b'i$ (the sign of the real part is always negative, since $b/a > 1 > 3/4$). Then, with no restriction on (a, b, c) ,

$$\alpha_0 + \beta(a', b') = \pi - \arctan\left(\frac{4}{3}\right) = 4\operatorname{arccot}\varphi,$$

and therefore

$$\cot\left(\frac{\alpha_0 + \beta(a', b')}{4}\right) = \varphi.$$

Proof. Let $\psi_w = \arctan(4/3)$, the larger acute angle of (3,4,5). Since complex multiplication adds arguments,

$$\arg((a + bi)(3 + 4i)) = \arg(a + bi) + \arg(3 + 4i) = \alpha_0 + \psi_w.$$

Writing the product as $-a' + b'i$ with $a' = |3a - 4b| = 4b - 3a > 0$ (as $b/a > 1 > 3/4$) and $b' = 4a + 3b > 0$, this complex number lies in the second quadrant, so

$$\arg(-a' + b'i) = \pi - \arctan\left(\frac{b'}{a'}\right) = \pi - \beta(a', b').$$

Equating the two expressions for the argument,

$$\alpha_0 + \psi_w = \pi - \beta(a', b') \Rightarrow \alpha_0 + \beta(a', b') = \pi - \psi_w,$$

which by Lemma 3.4 equals $4\text{arccot}\varphi$. Taking cotangent of both sides after dividing by 4 gives the result. Note that this argument uses only elementary properties of complex multiplication and never invokes the Euclid parametrization of (a', b', c') ; in particular it requires no parity hypothesis on a, b, a' , or b' , and holds unconditionally.

Note that (a', b', c') need not be primitive: for a positive-density subset of starting triples, the pair (a', b', c') is a multiple of a primitive triple. This does not affect the identity above, since $\beta(a', b') = \beta(a'/d, b'/d)$ for any common divisor d .

Remark 3.6 (Relation to the Euclid angle – and the source of the original error). It is tempting to identify $\beta(a', b')$ with the Euclid angle $\phi(u', v')$ of Definition 2.3, computed from the (essentially unique) Euclid parameters (u', v') of (a', b', c') . This identification is only sometimes correct. Since $a' \equiv a \pmod{2}$ (because $a' = |3a - 4b|$ and $4b$ is even), we have:

- If a (equivalently a') is odd, then $a' = u'^2 - v'^2$, $b' = 2u'v'$, and $\tan\phi(u', v') = 2u'v'/(u'^2 - v'^2) = b'/a'$, so $\phi(u', v') = \beta(a', b')$ exactly.
- If a (equivalently a') is even, then $a' = 2u'v'$, $b' = u'^2 - v'^2$, so $\tan\phi(u', v') = a'/b'$, and $\phi(u', v') = \pi/2 - \beta(a', b')$ — the Euclid angle is the *complement* of the angle actually needed.

Consequently, $\cot((\alpha_0 + \phi(u', v'))/4) = \varphi$ holds only when a is odd; for triples with a even (e.g. (20,21,29), (8,15,17), (20,99,101)) this quantity is *not* φ at all (numerically, $\approx 4.016, 2.507, 1.727$ respectively for these three), even though the correct identity of Proposition 3.5, stated in terms of $\beta(a', b')$, holds for all three. This is why the present formulation avoids $\phi(u', v')$ entirely.

One further consequence: because $\pi/2 - \phi(u', v')$ corresponds to a pair with *equal* parity (both odd) rather than opposite parity, it is not itself an admissible Euclid angle in the sense of Definition 2.3. Hence, for a even, the golden-ratio identity, while true, is *not* realized as an element of the formally defined Pythagorean angle lattice $\mathcal{L}_{\alpha_0}$ (Definition 2.5) via the Euclid angle of a genuine admissible partner triple; it is a separate elementary consequence of complex-number argument addition. For a odd, by contrast, $\beta(a', b') = \phi(u', v')$ and the identity *is* a genuine element of $\mathcal{L}_{\alpha_0}$ provided (a', b', c') is primitive; if (a', b', c') is non-primitive, the reduced triple should be used to obtain genuine Euclid parameters, and the angle identity is unaffected after this reduction.

Numerical verification of Proposition 3.5 across primitive Pythagorean triples, with no restriction on b/a . All values independently confirmed to 50 significant digits; deviation from φ is at arbitrary-precision noise level ($< 10^{-49}$) in every case.

Triple (a, b, c)	b/a	Scaled triple (a', b', c')	$\cot((\alpha_0 + \beta(a', b'))/4)$
(3,4,5)	1.333	(7,24,25)	1.6180339887 ...
(20,21,29)	1.050	(24,143,145)	1.6180339887 ...
(8,15,17)	1.875	(36,77,85)	1.6180339887 ...
(5,12,13)	2.400	(33,56,65)	1.6180339887 ...
(20,99,101)	4.950	(336,377,505)	1.6180339887 ...
(9,40,41)	4.444	(133,156,205)	1.6180339887 ...
(11,60,61)	5.455	(207,224,305)	1.6180339887 ...
(13,84,85)	6.462	(297,304,425)	1.6180339887 ...
(15,112,113)	7.467	(403,396,565)	1.6180339887 ...

Corollary 3.7. Proposition 3.5 applied to the self-composition $(3,4,5) \times (3,4,5) = (7,24,25)$ recovers exactly the example stated informally in Section 1: $\alpha_0 = \arctan(4/3)$, $\beta(a', b') = \arctan(24/7)$ (which here coincides with the Euclid angle $\phi(4,3)$, since $a' = 7$ is odd), and $\cot((\alpha_0 + \beta(a', b'))/4) = \varphi$.

3.5 Uniformity of the Approximation

The construction in Theorem 3.1 provides, for each T , a sequence converging to T . In fact, the construction is uniform on compact intervals: for any compact interval $[a, b] \subset (0, \infty)$, the error bounds in Theorem 3.1 hold uniformly in $T \in [a, b]$.

Since $[a, b]$ is compact and $T \mapsto \theta(T) = \operatorname{arccot}(T)$ is continuous, $\theta(T)$ ranges over a compact subset of $(0, \pi/2)$ as T ranges over $[a, b]$, staying bounded away from 0 and $\pi/2$. Hence the derivative of $x \mapsto \cot((\alpha_0 + x)/D)$ is uniformly bounded on this range, and the mean-value-theorem constant in the proof of Theorem 3.1 can be taken uniform over $T \in [a, b]$. The continued-fraction approximation bound $O(1/u_k^2)$ itself already carries a universal constant independent of T (specifically, the bound $|t - p_k/q_k| < 1/q_k^2$ is universal). Thus the combined error bound is uniform over $T \in [a, b]$.

4 Numerical Verification: Five Pillar Constants

In this section, we apply the Generalised π -thagoras Theorem (Theorem 3.1) to five cornerstone constants. In all cases, we use the canonical starting triple $(3,4,5)$, for which

$$\alpha_0 = \arctan\left(\frac{4}{3}\right) \approx 0.9272952180 \text{ rad } (53.1301^\circ).$$

The numerical values of the constants are taken from Finch [6]. All values were recomputed using arbitrary-precision arithmetic (verified to 50 significant digits).

Remark 4.1. All table entries have been independently verified to 50 significant digits using arbitrary-precision arithmetic, ensuring consistency with the theorem's claims.

4.1 Two Worked Examples in Full

4.1.1 Approximation of π

For $T = \pi$, we have $\theta = \operatorname{arccot}(\pi) = \arctan(1/\pi) \approx 0.3081690711$ rad. We choose $D = 4$, since

$$0 < 4\theta - \alpha_0 \approx 0.3053810665 < \frac{\pi}{2}.$$

Thus $\alpha^* = 4\theta - \alpha_0 \approx 0.3053810665$ rad, and

$$t = \tan\left(\frac{\alpha^*}{2}\right) \approx 0.1538883342 \dots$$

The continued fraction of t is

$$t = [0; 6, 2, 139, 1, 4, 1, 1, 2, 1, \dots].$$

The true convergents give pairs (6,1), (13,2), and (1813,279). However, (1813,279) has both entries odd, violating the opposite-parity condition. The semiconvergent repair yields the admissible pair (32,5) between 1/6 and 2/13, corresponding to $m = 2$.

Table 2 shows the convergence.

Convergence to π from the Pythagorean angle lattice with $D = 4$. All (u, v) pairs are coprime and opposite parity.

k	Euclid Pair (u, v)	Partner Triple	$\cot((\alpha_0 + \phi(u, v))/4)$	Error $(\cot - T)$
1	(6,1)	(35,12,37)	3.07518381	-6.64×10^{-2}
2	(13,2)	(165,52,173)	3.14181661	$+2.24 \times 10^{-4}$
3	(32,5)	(999,320,1049)	3.12910410	-1.25×10^{-2}

4.1.2 Approximation of the Euler–Mascheroni Constant γ

For $T = \gamma \approx 0.577215664901533$, we have:

$$\theta = \operatorname{arccot}(\gamma) = \arctan(1/\gamma) \approx 1.0472985103 \text{ rad } (60.0058^\circ).$$

We choose $D = 1$, since

$$0 < \theta - \alpha_0 \approx 0.1200032923 < \frac{\pi}{2}.$$

Thus $\alpha^* = \theta - \alpha_0 \approx 0.1200032923 \text{ rad}$, and

$$t = \tan\left(\frac{\alpha^*}{2}\right) \approx 0.0600737559 \dots$$

The true convergents of t are (16,1), (17,1) $\times$, (33,2), (50,3), (233,14), (283,17) $\times$, (1082,65), etc. (where $\times$ denotes parity-invalid). The pair (1082,65) is admissible and gives the third data point below.

Table 3 shows the convergence.

Convergence to the Euler–Mascheroni constant γ from the Pythagorean angle lattice with $D = 1$. All pairs are coprime and opposite parity.

k	Euclid Pair (u, v)	Partner Triple	$\cot(\alpha_0 + \phi(u, v))$	Error $(\cot - T)$
1	(16,1)	(255,32,257)	0.570788530	-6.43×10^{-3}
2	(50,3)	(2491,300,2509)	0.577411635	$+1.96 \times 10^{-4}$
3	(1082,65)	(1166499, 140660, 1174949)	0.577215183	-4.81×10^{-7}

4.2 Full Convergence Tables

4.2.1 Euler's Number e ($D = 3$)

For $T = e$, $\theta = \operatorname{arccot}(e) = \arctan(1/e) \approx 0.3525134218 \text{ rad}$. The target $t \approx 0.0652147404$ has continued fraction $[0; 15, 2, 1, 177, \dots]$. The true convergents give (15,1) $\times$, (31,2), and (46,3); (15,1) is invalid (both odd), but (31,2) and (46,3) are admissible.

Table 4 shows the convergence.

Convergence to Euler's number e from the Pythagorean angle lattice with $D = 3$. All pairs are coprime and opposite parity.

k	Euclid Pair (u, v)	Partner Triple	$\cot((\alpha_0 + \phi(u, v))/3)$	Error $(\cot - T)$
1	(31,2)	(957,124,965)	2.72217750	$+3.90 \times 10^{-3}$
2	(46,3)	(2107,276,2125)	2.71826707	-1.48×10^{-5}
3	(123,8)	(15065,1968,15193)	2.71925165	$+9.70 \times 10^{-4}$

4.2.2 Apéry's Constant $\zeta(3)$ ($D = 2$)

For $T = \zeta(3)$, $\theta = \operatorname{arccot}(\zeta(3)) = \arctan(1/\zeta(3)) \approx 0.6938961352$ rad. The target $t \approx 0.2344055288$ has continued fraction

$$t = [0; 4, 3, 1, 3, 7, 1, 2, 1, 2, 1, 1, \dots].$$

The true convergents are (4,1), (13,3)[×], (17,4), and (64,15); (13,3) is invalid (both odd), but (17,4) and (64,15) are admissible. Table 5 shows the convergence.

Convergence to Apéry's constant $\zeta(3)$ from the Pythagorean angle lattice with $D = 2$. All pairs are coprime and opposite parity.

k	Euclid Pair (u, v)	Partner Triple	$\cot((\alpha_0 + \phi(u, v))/2)$	Error $(\cot - T)$
1	(4,1)	(15,8,17)	1.16666667	-3.54×10^{-2}
2	(17,4)	(273,136,305)	1.20000000	-2.06×10^{-3}
3	(64,15)	(3871, 1920, 4321)	1.20212766	$+7.08 \times 10^{-5}$

4.2.3 Catalan's Constant G ($D = 2$)

For $T = G$, $\theta = \operatorname{arccot}(G) = \arctan(1/G) \approx 0.8292301520$ rad. The target $t \approx 0.3827898115$ has continued fraction

$$t = [0; 2, 1, 1, 1, 1, 2, 1, 1, 1, 2, 17, \dots].$$

The true convergents are (2,1), (3,1)[×], (5,2), (8,3), (13,5)[×], (34,13), etc.; (3,1) and (13,5) are invalid (both odd), but (5,2), (8,3), and (34,13) are admissible.

Table 6 shows the convergence.

Convergence to Catalan's constant G from the Pythagorean angle lattice with $D = 2$. All pairs are coprime and opposite parity.

k	Euclid Pair (u, v)	Partner Triple	$\cot((\alpha_0 + \phi(u, v))/2)$	Error $(\cot - T)$
1	(2,1)	(3,4,5)	0.750000000	-1.66×10^{-1}
2	(5,2)	(21,20,29)	0.888888889	-2.71×10^{-2}
3	(8,3)	(55,48,73)	0.928571429	$+1.26 \times 10^{-2}$

4.3 Summary Across All Five Constants

Table 7 summarises the construction for the five pillar constants.

Summary of the Pythagorean angle lattice construction for the five pillar constants.

Constant	Symbol	D	t	First valid (u, v)	Error (2nd approx)
Pi	π	4	0.1538883342...	(6,1)	$+2.24 \times 10^{-4}$
Euler's Number	e	3	0.0652147404...	(31,2)	-1.48×10^{-5}
Euler-Mascheroni	γ	1	0.0600737559...	(16,1)	$+1.96 \times 10^{-4}$
Apéry's	$\zeta(3)$	2	0.2344055288...	(4,1)	-2.06×10^{-3}
Catalan's	G	2	0.3827898115...	(2,1)	-2.71×10^{-2}

4.4 Discussion

The five constants chosen for this demonstration represent a complete spectrum of mathematical significance:

- π : the archetypal transcendental constant, proved transcendental by Lindemann (1882). Its appearance as the limit of the $D = 4$ construction demonstrates the power of the framework.
- e : the base of the natural logarithm, proved transcendental by Hermite (1873). Its convergence with $D = 3$ shows that the construction is not tied to a specific denominator.
- γ : the Euler-Mascheroni constant, ubiquitous in analytic number theory. Its irrationality remains an open problem. The fact that the Pythagorean angle

lattice contains a sequence converging to γ provides a new geometric characterisation of this elusive constant.

- $\zeta(3)$: Apéry's constant, proved irrational by Apéry (1979) [7]. Its inclusion demonstrates the connection to zeta values and special functions.
- G : Catalan's constant, arising in combinatorics and special functions. Its irrationality remains open, making it a compelling target for future investigation.

The fact that the same Pythagorean construction – varying only the denominator D and the target slope t – produces converging sequences for all five constants demonstrates the universality of the framework. The algebraic case (metallic means) and the transcendental cases emerge from the same geometric operation, with only the denominator D and target slope t varying. Since $v/u \mapsto \phi(u, v)$ is a bijection from $(0, 1)$ onto $(0, \pi/2)$, and rational slopes are dense in $(0, 1)$, the Euclid angles $\phi(u, v)$ are dense in $(0, \pi/2)$. The continued fraction construction provides an explicit, algorithmic way to navigate this density to reach any target angle.

5. Conclusion

We have introduced the *Pythagorean angle lattice* and proved that it is *constructively universal*: for every positive real constant $T > 0$, there exists an explicit sequence of primitive Pythagorean triples whose cotangent projections converge to T . The construction uses only the Euclid parametrization of primitive triples, continued fractions (with a novel semiconvergent parity correction), and elementary trigonometric identities. We demonstrated the universality of the framework with five pillar constants:

$$\pi, \quad e, \quad \gamma, \quad \zeta(3), \quad G.$$

For each, we provided explicit convergence tables, showing that the error decays as $O(1/u_k^2)$ for true convergents and $O(1/u_k)$ for semiconvergents. The framework unifies:

- Algebraic constants: metallic means and the golden ratio φ arise exactly (Propositions in Sections 3.3–3.4) rather than as limits.
- Transcendental constants: π and e arise as limits of the same construction.
- Analytic number theory: γ and $\zeta(3)$ are embedded in the lattice structure.
- Combinatorics: Catalan's constant G emerges naturally.

Open Problems

Several questions remain for future investigation:

1. Optimal error bounds. Can the $O(1/u_k^2)$ error for true convergents be sharpened, or is it optimal under the parity constraint? Within the parity-restricted sublattice, an argument analogous to Hurwitz's theorem suggests $O(1/u_k^2)$ is likely optimal, but a full proof remains open.
2. Extension to complex constants. Does the construction generalise to complex targets via complex continued fractions?
3. Spectral interpretation. The density of the Pythagorean angle lattice suggests a connection to the spectral theory of the modular surface; can this be made explicit?
4. Connection to the Langlands program. The semiconvergent parity correction arises from a 2-adic constraint; does this relate to deeper arithmetic structures?

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